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Riccati equations and convolution formulae for functions of Rayleigh type

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Abstract. Kishore (1963 *Proc. Am. Math. Soc.* **14** 527) considered the Rayleigh functions $\sigma_n(\nu) = \sum_{k=1}^{\infty} j_{\nu k}^{-2n}$, $n = 1, 2, \ldots$, where $\pm j_{\nu k}$ are the (non-zero) zeros of the Bessel function $J_\nu(z)$ and provided a convolution-type sum formula for finding σ_n in terms of $\sigma_1, \ldots, \sigma_{n-1}$. His main tool was the recurrence relation for Bessel functions. Here we extend this result to a larger class of functions by using Riccati differential equations. We get new results for the zeros of certain combinations of Bessel functions and their first and second derivatives as well as recovering some results of Buchholz for zeros of confluent hypergeometric functions.

1. Introduction

The Rayleigh functions are defined, for example, in [1, p 502], by the formula

$$\sigma_n(\nu) = \sum_{k=1}^{\infty} j_{\nu k}^{-2n} \qquad n = 1, 2, \ldots \tag{1}$$

where $\pm j_{\nu k}$ are the zeros of the Bessel function

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\,\Gamma(\nu+n+1)}. \tag{2}$$

They form the basis of an old method due to Euler, Rayleigh and others for evaluating the zeros. For example, in the case $\nu > -1$, the inequalities

$$[\sigma_n(\nu)]^{-1/n} < j_{\nu 1}^2 < \sigma_n(\nu)/\sigma_{n+1}(\nu) \qquad n = 1, 2, \ldots$$

provide infinite sequences of successively improving upper and lower bounds for $j_{\nu 1}^2$. Several authors have considered the question of finding 'sum rules' or formulae for $\sigma_n(\nu)$. By a method originating with Euler (see [1, pp 500ff] for details; various ramifications were considered recently in [2]), we can find all the $\sigma_n(\nu)$ in terms of the coefficients in the series (2). If we want to deal (as in [3]) with properties of the $\sigma_n(\nu)$ as functions of ν, there is a useful compact convolution formula due to Kishore [4]

$$\sigma_n(\nu) = \frac{1}{\nu+n} \sum_{k=1}^{n-1} \sigma_k(\nu)\,\sigma_{n-k}(\nu) \tag{3}$$

from which the $\sigma_n(\nu)$ may be found successively, starting from

$$\sigma_1(\nu) = 1/[4(\nu+1)]. \tag{4}$$

The question arises as to whether there are Kishore-type formulae for sums of zeros of other special functions such as the first and second derivatives of the Bessel function. In [5] there is a variant of this result for the zeros of the more general function

$$N_\nu(z) = az^2 J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z) \tag{5}$$

considered by Mercer [6]. The result of [5] gave a method of finding the reciprocal power sums

$$\tau_n(\nu) = \sum_{k=1}^{\infty} x_{\nu k}^{-2n} \qquad n = 1, 2, \ldots \tag{6}$$

where $x_{\nu k}$ are the zeros of the function $N_\nu(z)$. The main result of [5] expressed τ_n in terms of $\tau_k, k = 1, \ldots, n-1$ *and* $\sigma_k, k = 1, \ldots, n$. It seems desirable to express τ_n in terms of $\tau_k, k = 1, \ldots, n-1$ only. We do this here by using the Riccati equation satisfied by $z^{-\nu/2}N_\nu(x^{1/2})$. We also record the second-order linear differential equations satisfied by $N_\nu(z)$ and by $z^{-\nu/2}N_\nu(z^{1/2})$ since these do not seem to appear in the literature and may prove useful for other purposes.

The functions $\sigma_n(\nu)$ and $\tau_n(\nu)$ can be extended to non-integer and even complex values of n, providing generalized zeta functions. The σ_n case has been dealt with frequently [7]; the idea has even been extended to zeros of q-Bessel functions [8]. Here we confine our attention to the case of positive integral n.

In section 4, we apply our method to obtain power sums for zeros of confluent hypergeometric functions.

2. Differential equations for functions related to Bessel functions

The Bessel function $y = J_\nu(z)$ satisfies the differential equation

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0 \tag{7}$$

and the function $y = zJ_\nu'(z) + cJ_\nu(z)$ satisfies [9, p 13] the differential equation

$$z^2(z^2 - \nu^2 + c^2)y'' - z(z^2 + \nu^2 - c^2)y' + [(z^2 - \nu^2)^2 + 2cz^2 + c^2(z^2 - \nu^2)]y = 0.$$

Here we record the more general second-order linear differential equation satisfied by the function

$$Y = N_\nu(z) = az^2J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z). \tag{8}$$

It is

$$z^2Y'' + A(z)zY' + [B(z) + z^2 - \nu^2]Y = 0 \tag{9}$$

where

$$A(z) = \frac{-3a^2z^4 + pz^2 + q}{a^2z^4 - pz^2 + q}$$

$$B(z) = \frac{2a(a+b)z^4 + 2rz^2}{a^2z^4 - pz^2 + q}$$

with

$$p = 2a(a\nu^2 + c) + (a^2 - b^2)$$

$$q = (a\nu^2 + c)^2 - \nu^2(a-b)^2$$

and

$$r = a\nu^2(3a - b) + c(a + b).$$

We found equation (9) by repeated use of

$$zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z) \tag{10}$$

to express the derivatives $J_\nu^{(n)}(z), n = 1, 2, \ldots$ in terms of $J_\nu(z)$, $J_{\nu+1}(z)$ and discovered an appropriate vanishing linear combination of $N_\nu(z)$, $N'_\nu(z)$ and $N''_\nu(z)$. Of course, once (9) is known, it is easy to verify that $N_\nu(z)$, given by (8), satisfies it.

It is convenient to consider the function

$$y_\nu(z) = z^{-\nu/2}N_\nu(z^{1/2}) \tag{11}$$

where we choose that branch of $z^{1/2}$ which is positive for $z > 0$. Using (9), we find that the function $y_\nu(z)$ satisfies

$$4t^2\frac{\mathrm{d}^2y_\nu}{\mathrm{d}t^2} + [4\nu + 2 + 2A(t^{1/2})]t\frac{\mathrm{d}y_\nu}{\mathrm{d}t} + [t - \nu + \nu A(t^{1/2}) + B(t^{1/2})]y_\nu = 0. \tag{12}$$

It is well known that if y satisfies

$$y'' + P(t)y' + Q(t)y = 0 \tag{13}$$

then $u = y'/y$ satisfies the Riccati equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + P(t)u + Q(t) + u^2 = 0. \tag{14}$$

Applying this to (12), we find that, with $y_\nu(z)$ given by (8), $u = y'_\nu(z)/y_\nu(z)$ satisfies

$$\begin{aligned}4t(a^2t^2 - pt + q)\left[\frac{\mathrm{d}u}{\mathrm{d}t} + u^2\right] + 4[a^2(\nu - 1)t^2 - \nu pt + q(\nu + 1)]u + a^2t^2\\ +[p + 4a^2\nu - 2a(a + b)]t + 2\nu p + q + 2r = 0.\end{aligned} \tag{15}$$

3. Functions of Rayleigh type

The even entire function $z^{-\nu}N_\nu(z)$ has an infinite set of zeros $\pm t_n, n = 1, 2, \ldots$ with

$$\sum |t_k^{-2}| < \infty$$

so the zeros of $y_\nu(z)$ are $\zeta_k = t_k^2$, with

$$\sum |\zeta_k^{-1}| < \infty.$$

Thus

$$y_\nu(z) = z^{-\nu/2}N_\nu(z^{1/2}) = \frac{a\nu^2 + c + (b - a)\nu}{2^\nu\Gamma(\nu + 1)}\prod_{k=1}^{\infty}\left(1 - \frac{z}{\zeta_k}\right). \tag{16}$$

The constant multiplicative factor is obtained from the series (2). The validity of this infinite product expansion follows from facts on entire functions of finite order [10, chapter 8]. See [2] for a more complete discussion, with references, on the zeros of $N_\nu(z)$.

We may differentiate (16) logarithmically [11], to obtain

$$\frac{y'_\nu(z)}{y_\nu(z)} = -\sum_{k=1}^{\infty}\frac{1/\zeta_k}{1 - z/\zeta_k} = -\sum_{k=1}^{\infty}\frac{1}{\zeta_k}\sum_{n=0}^{\infty}\frac{z^n}{\zeta_k^n}.$$

This gives

$$2z\frac{y_\nu'(z)}{y_\nu(z)} = -2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} z^n/\zeta_k^n.$$

But we may interchange the order of summation here (since the iterated series converges absolutely) to obtain

$$2z\frac{y_\nu'(z)}{y_\nu(z)} = -2\sum_{n=1}^{\infty} z^n \sum_{k=1}^{\infty} \zeta_k^{-n} = -2\sum_{n=1}^{\infty} \tau_n z^n \tag{17}$$

where

$$\tau_n = \sum_{k=1}^{\infty} \zeta_k^{-n}. \tag{18}$$

Using

$$u = -\sum_{k=0}^{\infty} \tau_{k+1} z^k$$

we obtain

$$u^2 = \sum_{k=2}^{\infty}\left[\sum_{m=1}^{k-1} \tau_m \tau_{k-m}\right] z^{k-2}.$$

Substituting in (15), and comparing coefficients of powers of z, we obtain

$$\begin{aligned} \tau_1(\nu) &= \frac{2\nu p + q + 2r}{4q(\nu+1)} \\ 4q(\nu+2)\tau_2 &= 4q\tau_1^2 + 4\nu p\tau_1 - p - 4a^2\nu + 2a(a+b) \end{aligned} \tag{19}$$

$$4q(\nu+3)\tau_3 = 4p(\nu+1)\tau_2 - 4a^2(\nu-1)\tau_1 + a^2 + 8q\tau_1\tau_2 - 4p\tau_1^2 \tag{20}$$

and, for $k \geqslant 3$,

$$\begin{aligned} q(k+\nu+1)\tau_{k+1} &= p(k+\nu-1)\tau_k - a^2(k+\nu-3)\tau_{k-1} \\ &\quad + q\sum_{m=1}^{k} \tau_m \tau_{k-m+1} - p\sum_{m=1}^{k-1} \tau_m \tau_{k-m} + a^2 \sum_{m=1}^{k-2} \tau_m \tau_{k-m-1}. \end{aligned} \tag{21}$$

In the special case $a = b = 0, c = 1$ (and hence $p = 0, q = 1, r = 0$), where we are dealing with the zeros of the Bessel function, these reduce, as they should, to (4) and the convolution formula (3) for $\sigma_n, n = 2, 3, \ldots$.

In the special case $a = c = 0, b = 1$ (and hence $p = -1, q = -\nu^2, r = 0$), we are dealing with the non-trivial zeros of the function $J_\nu'(z)$; (19)–(21) become

$$\begin{aligned} \tau_1 &= \frac{\nu+2}{4(\nu+1)\nu} \\ \tau_2 &= \frac{-4\nu^2\tau_1^2 - 4\nu\tau_1 + 1}{-4\nu^2(\nu+2)} \\ \nu^2(\nu+3)\tau_3 &= (\nu+1)\tau_2 + 2\nu^2\tau_1\tau_2 - \tau_1^2 \end{aligned} \tag{22}$$

and for $k \geqslant 3$,

$$-\nu^2(k+\nu+1)\tau_{k+1} = -(k+\nu-1)\tau_k - \nu^2\sum_{m=1}^{k-1} \tau_m \tau_{k-m+1} + \sum_{m=1}^{k-2} \tau_m \tau_{k-m}. \tag{23}$$

In particular, these lead to

$$\tau_2 = \sum_{k=1}^{\infty} [j'_{\nu k}]^{-4} = \frac{1}{16} \frac{\nu^2 + 8\nu + 8}{\nu^2(\nu+1)^2(\nu+2)} \tag{24}$$

$$\tau_3 = \sum_{k=1}^{\infty} [j'_{\nu k}]^{-6} = \frac{1}{32} \frac{\nu^3 + 16\nu^2 + 38\nu + 24}{\nu^3(\nu+1)^3(\nu+2)(\nu+3)} \tag{25}$$

the same results as are obtained by the power-series method in [2].

4. Confluent hypergeometric functions

Buchholz [12] studied the non-trivial zeros a_λ of the function

$$M_{\kappa,\mu/2}(z) = \frac{z^{b/2} \mathrm{e}^{-z/2}}{\Gamma(1+\mu)} \, {}_1F_1(a; b; z) \tag{26}$$

and showed that these zeros are all simple and that there are infinitely many of them in the case where $a \neq -n$. He considered

$$S_p = \sum_{\lambda=1}^{\infty} a_\lambda^{-p}$$

and showed that it converges for all $p > 1$ but that it is divergent for $p \leqslant 1$.

He also gave explicit formulae for $S_2, \ldots, S_6$ and a method (far from explicit) for expressing S_{k+1} as a linear combination of $S_2, \ldots, S_{k-1}$. In (34) below we give a convolution formula for this task.

The function $w = {}_1F_1(a; b; z)$ satisfies

$$zw'' + (b - z)w' - aw = 0 \tag{27}$$

so $u = w'/w$ satisfies the Riccati equation

$$zu' + (b - z)u - a + zu^2 = 0. \tag{28}$$

From the Weierstrass product representation theorem, we obtain

$$w = \mathrm{e}^{az/b} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \mathrm{e}^{z/z_k}. \tag{29}$$

Differentiating (29) logarithmically [11],

$$\begin{aligned} u(z) = \frac{w'(z)}{w(z)} &= \frac{a}{b} - \sum_{k=1}^{\infty} \left[\frac{1/z_k}{1 - z/z_k} - \frac{1}{z_k} \right] \\ &= \frac{a}{b} - \sum_{k=1}^{\infty} \frac{1}{z_k} \left\{ \left[1 - \frac{z}{z_k} \right]^{-1} - 1 \right\} \\ &= \frac{a}{b} - \sum_{k=1}^{\infty} S_{k+1} z^k \end{aligned} \tag{30}$$

where the interchange of orders of summation here is justified by the absolute convergence of the iterated series. From this we have

$$zu'(z) = -\sum_{k=1}^{\infty} k S_{k+1} z^k \tag{31}$$

and

$$[u(z)]^2 = (a/b)^2 - 2(a/b)\sum_{k=1}^{\infty} S_{k+1}z^k + \sum_{k=2}^{\infty}\left(\sum_{m=1}^{k-1} S_{m+1}S_{k-m+1}\right)z^k. \tag{32}$$

Thus equation (28) becomes

$$-\sum_{k=1}^{\infty}(b+k)S_{k+1}z^k + \left[1-\frac{2a}{b}\right]\sum_{k=1}^{\infty} S_{k+1}z^{k+1} + \left[\frac{a^2}{b^2}-\frac{a}{b}\right]z + \sum_{k=2}^{\infty}\left(\sum_{m=2}^{k} S_m S_{k-m+2}\right)z^{k+1} = 0. \tag{33}$$

Comparing the coefficients of z^k, $k = 1, 2, \ldots$ in (33) we obtain:

$$\begin{aligned} S_2 &= \frac{a(a-b)}{b^2(b+1)} \\ S_3 &= \frac{a(a-b)(b-2a)}{b^3(b+1)(b+2)} \\ S_{k+1} &= \frac{1}{b(k+b)}\left[(b-2a)S_k + b\sum_{m=2}^{k-1} S_m S_{k-m+1}\right] \qquad k = 3, 4, \ldots. \end{aligned} \tag{34}$$

This leads, in particular, to

$$S_4 = \frac{a(a-b)[a(a-b)(5b+6)+b^2(b+1)]}{b^4(b+1)^2(b+2)(b+3)}$$

etc, agreeing with the results found by Buchholz [12].

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